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SYSTEMATIC EFFECT AS A PART OF THE COVERAGE INTERVAL

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Abstract

The paper concerns the problem of treatment of the systematic effect as a part of the coverage interval associated with the measurement result. In this case the known systematic effect is not corrected for but instead is treated as an uncertainty component. This effect is characterized by two components: systematic and random. The systematic component is estimated by the bias and the random component is estimated by the uncertainty associated with the bias. Taking into consideration these two components, a random variable can be created with zero expectation and standard deviation calculated by randomizing the systematic effect. The method of randomization of the systematic effect is based on a flatten-Gaussian distribution. The standard uncertainty, being the basic parameter of the systematic effect, may be calculated with a simple mathematical formula. The presented evaluation of uncertainty is more rational than those with the use of other methods. It is useful in practical metrological applications.

Keywords: measurement uncertainty, coverage interval, systematic effect, randomization.

1. Introduction

A systematic effect having two components, systematic and random [1], may be treated as a part of the coverage interval. In this case, the known systematic effect is not corrected for but instead is treated as an uncertainty contribution. The first component of the systematic effect is estimated by the bias and the second component is estimated by uncertainty associated with this bias. The new random variable can be created with zero expectation and the calculated standard deviation bases on the information connected with the bias and its own uncertainty.

2. Coverage interval

The concept of a coverage interval is connected with the probability distribution of the possible values for the measurand. In general, the statistical coverage interval is defined as "an interval for which it can be stated with a given level of confidence that it contains at least a specified proportion of the population" [1]. A specific definition in metrology of the coverage interval is "an interval containing the value of a quantity with a stated probability" [2]. There are two options: probabilistically symmetric coverage interval and shortest coverage interval. In case of symmetrical distribution of the possible values for the measurand there is only one interval, symmetric around its expectation:

$$I_p = [y_{\text{low}}, y_{\text{high}}], \tag{1}$$

where y_{low} and y_{high} are the endpoints of the coverage interval corresponding to the values $G^{-1}(\alpha)$ and $G^{-1}(\alpha + p)$, which are the α and $\alpha + p$ quantiles of distribution function $G(\eta)$ of the measurand. The usual assumption of $\alpha = 2{,}5$ % and $p = 95$ % gives the coverage interval

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defined by the 0,025 and 0,975 quantiles. With reference to the classical expression of expanded uncertainty U it can be written as:

$$I_p = [\,\bar{y} - U,\ \ \bar{y} + U]. \tag{2}$$

Satisfying:

$$\int_{\bar{y}-U}^{\bar{y}+U} g(\eta)\,\mathrm{d}\eta = p, \tag{3}$$

where $g(\eta)$ is a probability density function of the measurand, $\bar{y}$ is its estimate and p is a coverage probability.

3. Systematic effect as a random variable

The systematic effect contains the bias e, as the estimate of systematic error, and its standard uncertainty $u(e)$. We assume that the probability attributed to the random component of the effect is a normal distribution, and we assume $k = 2$ corresponding to a coverage probability p = 95 % (Fig. 1). Creating a new random variable with zero expectation we can determine a symmetrical coverage interval

$$U = |e| + 2 \cdot u(e). \tag{4}$$

Thus, defined U gives the expanded uncertainty of randomized systematic effect. The distribution of this random variable is a R∗N distribution.

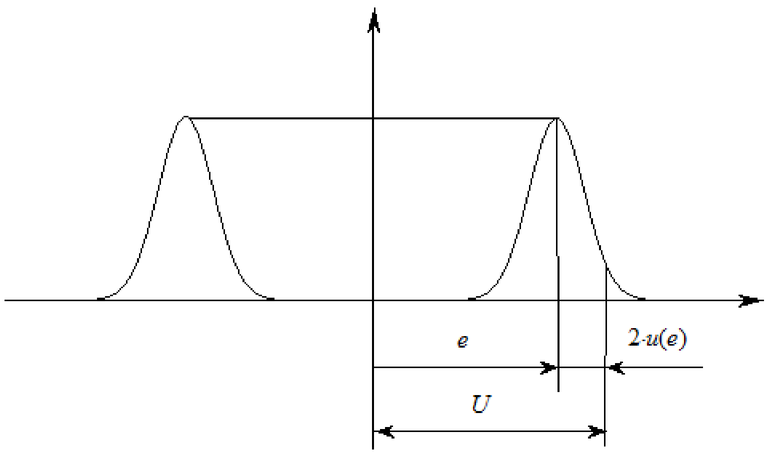

Fig. 1. Randomization of systematic effect.

4. R∗N distribution

The R∗N distribution is a convolution of two distributions, rectangular and normal. The probability density function of R∗N distribution is given by:

$$g_{\mathrm{RN}}(\eta) = \frac{1}{2\sqrt{6\pi}\, r} \int_{\eta-\sqrt{3}r}^{\eta+\sqrt{3}r} \exp\left(-\frac{\xi^2}{2}\right) \mathrm{d}\xi. \tag{5}$$

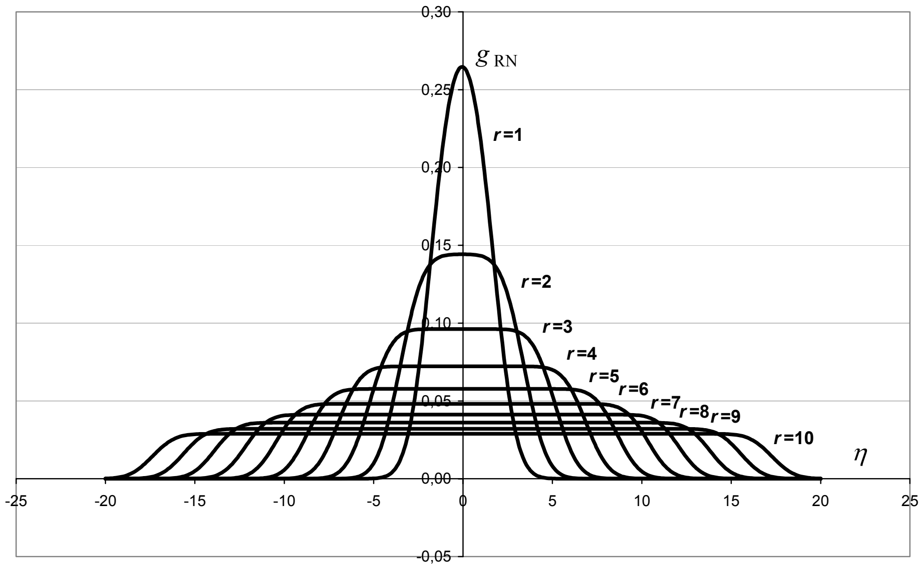

Fig. 2. Probability density function for convolution of rectangular and normal distributions with different values for parameter *r*.

The probability density functions of R∗N distributions are characterizing, in general, a constant value surrounding expectation and its slopes are Gaussian functions (Fig. 2). Therefore, sometimes this distribution is called a flatten-Gaussian distribution [3]. The range of constancy of the probability density function depends on parameter *r*, that is a ratio of the standard deviation σ_{R} of a rectangular distribution to the standard deviation σ_{N} of a normal distribution:

$$r = \frac{\sigma_{\mathrm{R}}}{\sigma_{\mathrm{N}}}. \tag{6}$$

The parameter *r* of R∗N distribution may be estimated by the formula connecting the bias and the standard uncertainty associated with this bias:

$$r_u = \frac{2 \cdot |e|}{3 \cdot u(e)} + 1. \tag{7}$$

For simple convolution of the rectangular and normal distributions, where $\sigma_{\mathrm{R}} = |e| / \sqrt{3}$ and $\sigma_{\mathrm{N}} = u(e)$, it is:

$$r = \frac{|e|}{\sqrt{3}\, u(e)}. \tag{8}$$

The above formulas are different, but formula (7) better approximates the parameter of randomized systematic effect and may be also used to characterize the R∗N distribution.

The coverage factor for the R∗N distribution should be calculated numerically. The coverage factor values corresponding to the coverage probability p = 95 % are presented in Table 1 [4–7]. The coverage factor can also be calculated as for the trapezoidal distribution from the formula [8–10]:

$$k_{\mathrm{T}} = \sqrt{\frac{3}{r_u^2 + 1}} \left(1 + r_u - 2\sqrt{r_u (1 - p)} \right). \tag{9}$$

The difference between coverage factor values calculated for the R∗N distribution and the trapezoidal distribution corresponding to coverage probability 95 % are presented in Tab. 2. For each relationship between e and $u(e)$ the differences are minimum.

Table 1. Coverage factor k_{RN} corresponding to coverage probability p = 95 % for limits of ratio r_u.

k_{RN}	r_u up to value	k_{RN}	r_u up to value	k_{RN}	r_u up to value
1,96	0,5090	1,85	1,6410	1,74	3,1930
1,95	0,6985	1,84	1,7380	1,73	3,4410
1,94	0,8240	1,83	1,8390	1,72	3,7300
1,93	0,9280	1,82	1,9460	1,71	4,0740
1,92	1,0220	1,81	2,0600	1,70	4,4925
1,91	1,1110	1,80	2,1820	1,69	5,0235
1,90	1,1980	1,79	2,3135	1,68	5,7350
1,89	1,2840	1,78	2,4560	1,67	6,7760
1,88	1,3700	1,77	2,6120	1,66	8,5975
1,87	1,4580	1,76	2,7845	1,65	∞
1,86	1,5480	1,75	2,9765		

Table 2. Standard uncertainty and coverage factor of randomized systematic effect.

$e/u(e)$	r_u	k_{RN}	k_T	U	$u_R=U/k_{RN}$	$u_R=U/k_T$	$u_{R(MCM)}$
0,1	1,0667	1,91	1,90	2,1	1,10	1,10	1,10
0,2	1,1333	1,90	1,90	2,2	1,16	1,16	1,15
0,3	1,2000	1,89	1,90	2,3	1,22	1,21	1,21
0,4	1,2667	1,89	1,89	2,4	1,27	1,27	1,28
0,5	1,3333	1,88	1,89	2,5	1,33	1,32	1,32
0,6	1,4000	1,87	1,88	2,6	1,39	1,38	1,39
0,7	1,4667	1,86	1,88	2,7	1,45	1,44	1,44
0,8	1,5333	1,86	1,87	2,8	1,51	1,49	1,50
0,9	1,6000	1,85	1,87	2,9	1,57	1,55	1,56
1	1,6667	1,84	1,86	3,0	1,63	1,61	1,62
2	2,3333	1,78	1,81	4,0	2,25	2,21	2,23
3	3,0000	1,74	1,77	5,0	2,87	2,83	2,85
4	3,6667	1,72	1,74	6,0	3,49	3,46	3,47
5	4,3333	1,70	1,71	7,0	4,12	4,08	4,10
6	5,0000	1,69	1,70	8,0	4,73	4,71	4,72
7	5,6667	1,68	1,69	9,0	5,36	5,34	5,35
8	6,3333	1,67	1,68	10	5,99	5,96	5,98
9	7,0000	1,66	1,67	11	6,63	6,59	6,61
10	7,6667	1,66	1,66	12	7,23	7,21	7,23

5. Standard uncertainty of the randomized systematic effect

The standard uncertainty of the randomized systematic effect is given as:

$$u_R = \frac{U}{k} = \frac{|e| + 2 \cdot u(e)}{k}, \tag{10}$$

where the coverage factor:

$$k = k_{RN} \approx k_T. \tag{11}$$

The differences between standard uncertainty values calculated for the randomized systematic effect with the use of the coverage factors k_{RN} and k_T are also presented in Table 2. This differences are minimum and do not influence the value of standard uncertainty as it is expressed with two significant digits. In Table 2 the values of standard uncertainty are presented with three significant digits to show the difference between them. The above-mentioned differences between them do not exceed two percent.

The standard uncertainty of the randomized systematic effect may also be calculated with the use of the Monte Carlo method. One can do computation from the formula [2]:

$$u^2_{\mathrm{R(MCM)}} = \frac{1}{M-1}\sum_{i=1}^{M}\left(y_i - \overline{y}\right)^2. \tag{12}$$

The values y_i are drawn from the R∗N distribution, having parameter $r = r_u$ given by equation (7). The results of $u_{\mathrm{R(MCM)}}$ computation are presented in Table 2. The differences between u_{R} values obtained by the analytical method and $u_{\mathrm{R(MCM)}}$ values obtained by the numerical method do not exceed one percent.

The random number generator of R∗N distribution may be built with the use of two simple random number generators. The random numbers are drawn through the formula:

$$y = \frac{r\,z_1 + z_2}{\sqrt{r^2+1}}, \tag{13}$$

where z_1 and z_2 are random variables having standardized rectangular distribution R(0, 1) and standardized normal distribution N(0, 1).

6. Comparison with literature approach

The approach presented in literature [11−13] most often gives the formula for calculating the standard uncertainty associated with the systematic effect:

$$u_{\mathrm{L}} = \sqrt{e^2 + u^2(e)}. \tag{14}$$

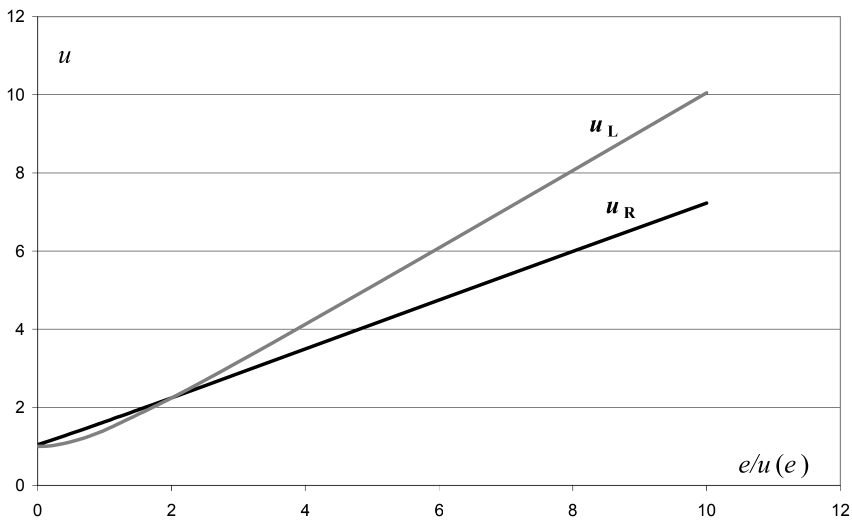

Fig. 3. Standard uncertainty of systematic effect calculated with two approaches.

In this approach the value of the bias e is treated as a standard uncertainty. Because the bias is always evaluated with a given uncertainty, the formula (14) also contains the standard uncertainty $u(e)$ associated with the estimation of e. This formula binds the systematic and random components of the systematic effect, expressing standard uncertainty u_L like the law of uncertainty propagation. When we calculate the uncertainty from this formula the value of u_L is a nonlinear function (Fig. 3).

In the approach presented here, the calculation of u_R creates practically a linear function. The growth of the value of the systematic effect component causes the proportional increase of standard uncertainty of this effect. Thus, the relation between the standard uncertainty, given by (10), and systematic effect components is practically linear.

7. Practical example

Practical use of the proposed randomization may be applied to interpretation of the information contained in the calibration certificate. In this certificate the bias is given with associated uncertainty. The reported expanded uncertainty of measurement is stated as the combined standard uncertainty multiplied by the coverage factor $k = 2$, which for a normal distribution corresponds to the coverage probability of approximately 95 %.

We can use a simple example concerning the measurement of a roller diameter by a calibrated micrometer. The calibration certificate of a measuring instrument states the bias in the whole measuring range is 3 μm with an associated uncertainty of 2 μm. We can assume that the bias is the estimate of the maximum systematic error and its absolute value may be equal or smaller than 3 μm for any measured diameter. In this case $|e| = 0{,}003$ mm and $u(e) =$ 0,001 mm, because expanded uncertainty $U = 0{,}002$ mm and $k = 2$. The randomizing R∗N distribution of that systematic effect has a parameter $r_u = 3$, given by formula (7). From Table 1 we can read $k_{RN} = 1{,}74$ or from the formula (9) we can calculate $k_T = 1{,}77$. The standard uncertainty, given by formula (10), is $u_R = 0{,}0029$ mm. In the case of trapezoidal approximation the standard uncertainty is $u_R = 0{,}0028$ mm, because a trapezoidal distribution has a smaller standard deviation than the R∗N distribution for the same parameter of a randomized quantity. This uncertainty we can also call type B and then it may be written $u_B = u_R$.

The roller diameter Φ20h7 (h7 is a symbol of diameter tolerance) was measured with an average of the observations $\bar{d} = 19{,}990$ mm, as the estimate of the diameter and with the experimental standard deviation of the mean $s(\bar{d}) = 0{,}0017$ mm, as the standard uncertainty. This uncertainty is called type A, then $u_A = s(\bar{d})$. In accordance with the law of uncertainty propagation the combined standard uncertainty is given as:

$$u_c(d) = \sqrt{u_A^2 + u_B^2}. \tag{15}$$

The combined standard uncertainty may be an estimate of standard uncertainty associated with the measurement result of the roller diameter obtained using a calibrated micrometer: $u(d) = u_c(d) = 0{,}0033$ mm. We can assume normal distribution attributed to the u_A uncertainty and the R∗N distribution attributed to the u_B uncertainty. The coverage interval may be calculated by the analytical method described in publications [4–6]. Using this method we can obtain: $d_{low} = 19{,}9838$ mm and $d_{high} = 19{,}9962$ mm. According to the recommendation of document [2] we can report the final result of measurement as:

$d = 19{,}9900$ mm, $u(d) = 0{,}0033$ mm
95 % coverage interval = [19,9838; 19,9962] mm

or we can present it in traditional form: d = (19,9900 ± 0,0062) mm. We can compare this result with the calculation using the Monte Carlo method [2]:

$$d = 19{,}9900 \text{ mm}, \; u(d) = 0{,}0034 \text{ mm}$$
$$95\ \% \text{ coverage interval} = [19{,}9837;\ 19{,}9963] \text{ mm}$$

or we can present it in traditional form: d = (19,9900 ± 0,0063) mm. The measurement result is the same when we round the standard uncertainty to one significant digit

$$d = 19{,}990 \text{ mm}, \; u(d) = 0{,}003 \text{ mm}$$
$$95\ \% \text{ coverage interval} = [19{,}984;\ 19{,}996] \text{ mm}$$

or express it in traditional form: d = (19,990 ± 0,006) mm. The tolerance of the diameter Φ20h7 is $T = 21$ µm with the upper specification limit: 0 µm and lower specification limit: -21 µm, corresponding to a maximum permissible diameter equal $d_{max} = 20$ mm and minimum permissible diameter equal $d_{min} = 19{,}979$ mm.

8. Conclusion

The systematic effect may be joined to the coverage interval of a measurement result. In this case the systematic effect is treated as an uncertainty component and a random variable. This random variable can be characterized by the R∗N distribution. The R∗N distribution covers two components of systematic effect, bias and uncertainty associated with this bias. The calculations of standard uncertainty and coverage factor of the randomized systematic effect are simple and can be easily implemented in practical application in metrology.

The literature approach does not assume the probability distribution for the systematic effect. Thus, the standard uncertainty associated with the systematic effect can be calculated only from the law of uncertainty propagation. The calculation presented above may be done by the analytical method as well as the numerical method with the use of propagation of distributions, recommended in [2]. Each method provides practically the same value of standard and expanded uncertainty.

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